Mathematics - Course 221

THE STRAIGHT LINE

I Slope of a Straight Line

The *slope* of a straight line in the xy-plane is a measure of how steeply the line rises or falls relative to the x-axis.

More precisely, the slope of a line is the increase in y per unit increase in x,

OR *the rate of change of y with respect to x.*

In Figure 1, for line segment P_1P_2,

$\Delta y = y_2 - y_1$ is called the *rise*

$\Delta x = x_2 - x_1$ is called the *run*, and

θ is called the *angle of inclination* of the line.

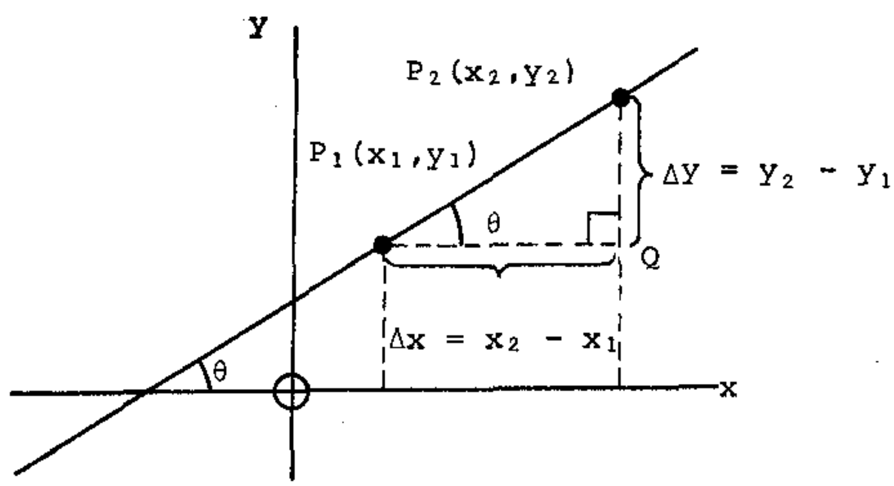

Figure 1

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The numerical value of the slope, usually designated "m", is given by

$$\boxed{\text{slope } m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}}$$

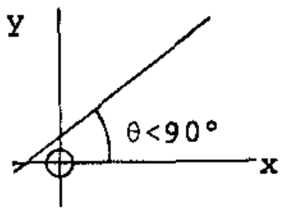
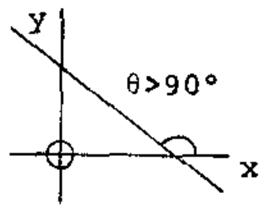
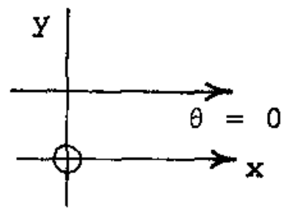
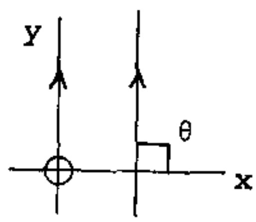
By trigonometry applied to right triangle P_1P_2Q of Figure 1,

$$\boxed{\tan \theta = \frac{\Delta y}{\Delta x} = m}$$

ie, the slope of a line is numerically equal to the tangent of the line's angle of inclination.

Note that the angle of inclination is defined as the smallest angle measured counterclockwise from the positive x-axis to the line, and therefore is always less than 180°.

The following table summarizes the correlation between the slope and orientation of a line in the plane:

Line Orientation	Typical Sketch	Slope Value
Rising to the right		$m > 0$
Falling to the right		$m < 0$
Parallel to x-axis		$m = 0$ $(\Delta y = 0)$
Perpendicular to x-axis		m undefined $(\Delta x = 0)$

Example 1

Find the (a) slope (b) angle of inclination of the line which passes through (-2,4) and (3,-5)

Solution

(a) $\text{Slope} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{-5 - 4}{3-(-2)}$

$= \dfrac{-9}{5}$ or -1.8

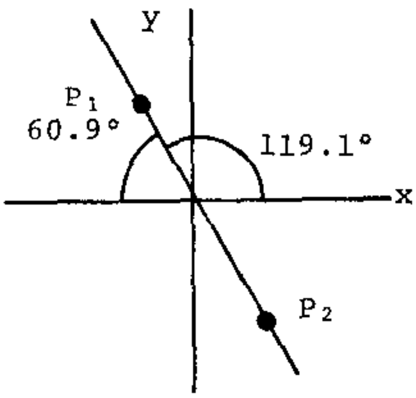
NOTE: In the previous solution, $P_1(x_1, y_1) = (-2, 4)$ and $P_2(x_2, y_2) = (3, -5)$. However, the choice for P_1 and P_2 could have been reversed without affecting the answer. (Check this.)

(b) $\tan\theta = -1.8$

$\Rightarrow$ associated acute angle $= \tan^{-1} 1.8$ (cf lesson 321.20-3)
$= 60.9°$

$\therefore$ angle of inclination, $= 180 - 60.9°$

$= \underline{\underline{119.1°}}$



Example 2

Given that the slope of a line is 1.5, find the change in

(a) x corresponding to an increase of 3 in y.

(b) y corresponding to a decrease of 4 in x.

Solution

Let $P(x,y)$ and $Q(x + \Delta x,\ y + \Delta y)$ be any two points on the line (see Figure 2).

Then slope of $PQ = \frac{\Delta y}{\Delta x} = 1.5$

(a) $\Delta y = 3 \Rightarrow \frac{3}{\Delta x} = 1.5$

ie, $\Delta x = \frac{3}{1.5}$

$= 2$

$\therefore$ x increases by 2 if y increases by 3 (between any two points on the line.)

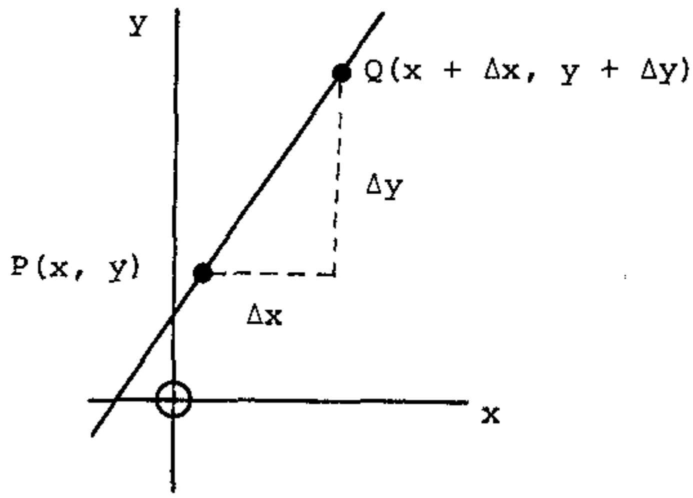


Figure 2

(b) $\Delta x = -4$ (x *in*creases by -4 if x *de*creases by 4).

Then $\frac{\Delta y}{-4} = 1.5$

$\therefore$ $\Delta y = (-4)(1.5)$
$= -6$

$\therefore$ y decreases by 6 if x decreases by 4.

II Parallel and Perpendicular Lines

(a) Parallel lines have equal slopes,

ie, line $L_1 \parallel$ line $L_2 \Leftrightarrow m_1 = m_2$

(b) The slopes of perpendicular lines are negative reciprocals,

ie, line $L_1 \perp$ line $L_2 \Leftrightarrow m_1 = -\frac{1}{m_2}$

Example 3

Find the slope of the family of lines (a) parallel (b) perpendicular to a line L with slope $m = \frac{2}{5}$.

Solution

(a) Slope of family of lines parallel to L $= m$

$$= \frac{2}{5}$$

(b) Slope of family of lines perpendicular to L $= -\frac{1}{m}$

$$= -\frac{1}{\frac{2}{5}}$$

$$= -\frac{5}{2}$$

III Equation of a Line

The *equation of a line* is the relationship which is satisfied by the coordinates of all points on the line, and by no others.

(a) Two-Point Form

Required: to find the equation of the line which passes through points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Solution: Let $P(x, y)$ be any point (other than P_1 or P_2) on the line (see Figure 3).

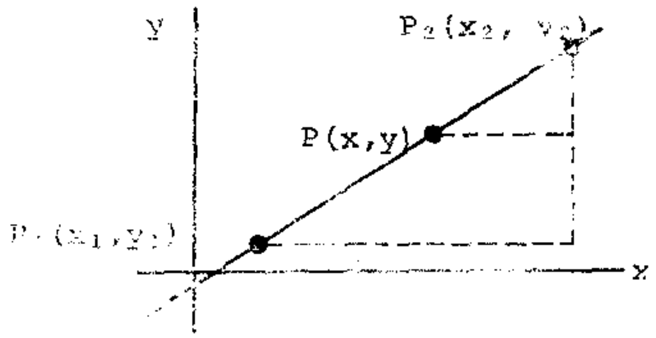


Figure 3

Then slope P_1P = slope P_1P_2 (all line segments have same slope)

ie, $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$$\therefore \quad \boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)} \qquad \text{Two-point form.}$$

Example 4

Find the equation of the line passing through points (-2,4) and (3,-5).

Solution: Using two-point form,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

ie, $y - 4 = \frac{-5 - 4}{3 - (-2)}(x - (-2))$

$$= \frac{-9}{5}(x + 2)$$

ie, $5y - 20 = -9x - 18$

ie, $\underline{\underline{9x + 5y - 2 = 0}}$

Note:

(i) The answer has been expressed in the so-called *general form* of the straight line equation, $Ax + By + C = 0$.

(ii) Points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ can be interchanged in the above solution without affecting the answer. (Check this.)

(b) Slope-Point Form

Required: to find the equation of the line having slope m and passing through $P_1(x_1, y_1)$.

Solution: Let $P(x,y)$ be any point on the line (see Figure 4).

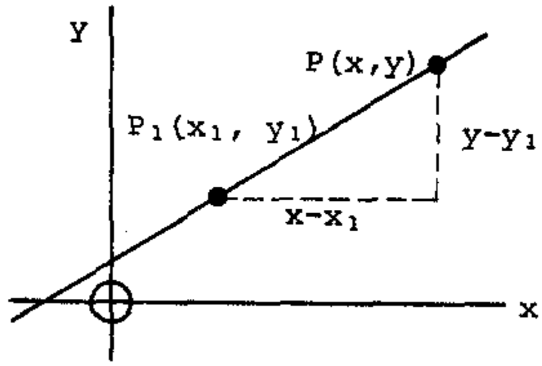


Figure 4

Then slope $P_1P = m$

ie, $\frac{y - y_1}{x - x_1} = m$

$\therefore$ $\boxed{y - y_1 = m(x - x_1)}$ Slope-Point Form.

Example 5

Find the equation of a line with slope -2 and passing through (-3,5).

Solution: Using slope-point form,

$y - y_1 = m(x - x_1)$

ie, $y - 5 = -2(x - (-3))$ (substitute (-3,5) for (x_1,y_1))

ie, $y - 5 = -2x - 6$

ie, $\underline{\underline{2x + y + 1 = 0}}$

(c) Slope-Intercept Form

Required: to find the equation of the line with slope **m** and y-intercept b.

Solution: Let P(x,y) be any point on the line (see Figure 5).

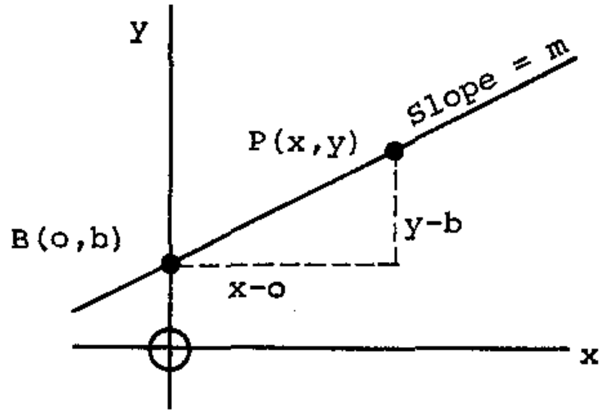


Figure 5

Then slope BP = m

ie, $\frac{y - b}{x - o} = m$

ie, $y - b = xm$

ie, $\boxed{y = mx + b}$ Slope-Intercept Form

Example 6

Find the equation of the line having slope $\frac{2}{3}$ and y-intercept -3.

Solution: Using slope-intercept form,

$y = mx + b$

ie, $y = \frac{2}{3}x + (-3)$

ie, $3y = 2x - 9$ (mult. both sides by 3)

ie, $\underline{\underline{2x - 3y - 9 = 0}}$

Example 7

Find the (a) slope (b) y-intercept (c) x-intercept of the line $5x - 2y + 10 = 0$.

Solution: The simplest way to find the slope and y-intercept is to express the equation in slope-intercept form by solving for y:

$$5x - 2y + 10 = 0$$

$$\therefore \quad -2y = -5x - 10$$

$$\therefore \quad y = \underset{\substack{\uparrow \\ m}}{\frac{5}{2}}x + \underset{\substack{\uparrow \\ b}}{5} \qquad (y = mx + b)$$

(a) slope $m = \frac{5}{2}$, and

(b) y-intercept $b = 5$

(c) At the x-intercept, $y = o$. Thus the x-coordinate is found by substituting $y = o$ in the equation, and solving for x:

$$5x - 2(0) + 10 = 0$$

$$\therefore \quad x = -2$$

$$\therefore \quad \text{x-intercept} = -2$$

<u>Example 8</u>

Find the equation of the line L_2 passing through the point $(-4,1)$, and perpendicular to line L_1 $3x - y - 2 = 0$.

<u>Solution</u>: Equation of L_1 in "$y = mx + b$" form is $y = 3x - 2$

$$\therefore m_1 = 3$$

$$\therefore m_2 = -\frac{1}{m_1}$$

$$= -\frac{1}{3}$$

$\therefore$ Equation of L_2 is $y - y_1 = m(x - x_1)$ (slope-point form)

$$y - 1 = -\frac{1}{3}(x - (-4)) \qquad ((x_1, y_1) = (-4,1))$$

$$\text{ie,} \quad 3y - 3 = -(x + 4)$$

$$= -x - 4$$

$$\therefore \quad \underline{\underline{x + 3y + 1 = 0}}$$

IV Graphing Lines

Recall that all equations of the form

$Ax + By + C = 0$ (general form) or

$y = mx + b$ (slope-intercept form),

represent straight lines in the xy-plane. The (x,y) co-ordinates of every point on a line (and no others) satisfy the equation of the line.

Steps to Graphing a Line

1. Solve the equation for y (or x).
2. Make a table of values containing at least three points. (The third point serves as an internal check: if all three points do not line up on graph, at least one point is in error.)
3. Plot points.
4. Draw and label line.

Example 9

Graph the line $2x - 5y + 6 = 0$

Step 1: $y = \frac{2x + 6}{5}$

Step 2:

x	-8	0	2
y	-2	$\frac{6}{5}$	2

Step 3, 4:

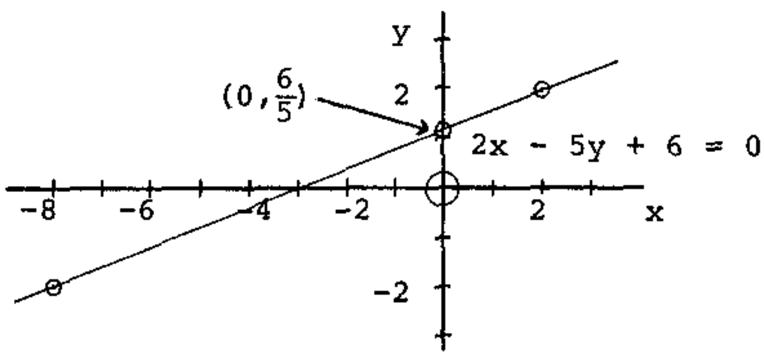


Figure 6

ASSIGNMENT

1. Find (i) the slope, (ii) the angle of inclination, and (iii) the equation the line passing through the points,

   (a) (0,0) and (3,4)

   (b) (0,2) and (3,0)

   (c) (2,-2) and (-2,2)

   (d) (5,2) and (0,2)

   (e) (-3,1) and (-3,4)

2. Show that the following three points lie on the same straight line:

   P(-5,-3), Q(-1,-1), R(5,2)

3. Graph the following lines and find their slopes and intercepts:

   (a) $x + y = 4$

   (b) $5x - 4y - 20 = 0$

   (c) $5y - 6 = 0$

   (d) $15x + 4 = 0$

4. State the slope of the family of lines (a) parallel (b) perpendicular to each of the lines in question 3.

5. Find the equation of the line passing through the given point with the given slope.

   (a) (4,3), $m = 1/3$

   (b) (-4,-1), $m = -5$

   (c) (-7,-5), $m = 0$

6. Find the equation of the line passing through the given point with the given angle of inclination.

   (a) $(3,3)$, $\theta = 45°$

   (b) $(-1,4)$, $\theta = 30°$

   (c) $(2,-5)$, $\theta = 135°$

7. Find the slope and y-intercept of each of the following lines:

   (a) $2x - 5y + 6 = 0$

   (b) $8x + 3y - 7 = 0$

8. For each line in question #7, state the change in

   (a) x corresponding to an increase of 3 in y.

   (b) y corresponding to a decrease of 5 in x.

9. Find the equations of the following lines:

   (a) passing through $(-1,4)$ and $(-1,-2)$

   (b) passing through $(-2,-5)$ with slope $\frac{5}{3}$

   (c) with y-intercept $-4\frac{1}{2}$ and slope $-\frac{2}{3}$

   (d) passing through $(0,0)$ and parallel to $4x + y - 2 = 0$

   (e) with y-intercept 6 and perpendicular to $x - 5y + 3 = 0$

   (f) passing through $(6,0)$ with angle of inclination $45°$.

L.C. Haacke